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LETTER TO THE EDITOR

Painlevé analysis and Bäcklund transformations of Doktorov–Vlasov equations

S Yu Sakovich

Institute of Physics, Academy of Sciences, F Skaryna Avenue 68, Minsk 72, Belarus

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Abstract. The singularity analysis of the system of nonlinear equations $\mathrm{i}a_t = a_{xx} + \alpha a^2 a^* - \mathrm{i}p$, $p_x + \mathrm{i}\beta p + ar = 0$, $r_x = \frac{1}{2}(ap^* + a^* p)$ (where * denotes the complex conjugation, functions a and p are complex, function r and constants α and β are real) indicates that the system has the Painlevé property at $\alpha = \frac{1}{2}$ only. This analytic exclusiveness of the case $\alpha = \frac{1}{2}$ agrees with results by Doktorov and Vlasov who selected the same case by a modification of the Wahlquist–Estabrook method and found a corresponding Lax pair. In the integrable case, the method of truncating Weiss–Tabor–Carnevale expansions determines a Bäcklund autotransformation which, unfortunately, violates the condition of complex conjugateness between a and a^*. Another Bäcklund autotransformation, compatible with this condition, is found by a technique of Miura transformations.

The Painlevé property, formulated for partial differential equations by Weiss, Tabor and Carnevale [1] a decade ago, is generally considered as a reliable sufficient condition for integrability of nonlinear systems [2]. Since any rigorous proof is absent, the whole confidence in the relation between the Painlevé property and integrability is based on results of the singularity analysis of particular integrable and non-integrable equations. An impressive confirmation of sufficiency of the Painlevé property for integrability is achieved in those investigations, where the singularity analysis is carried out for equations containing free parameters, and the Painlevé test selects integrable cases only [3–6]. In the present letter, devoted to the Painlevé analysis of a two-parameter nonlinear system of five equations of total order seven, the reliability of the Painlevé test will be confirmed again. Moreover, we will construct two *different* Bäcklund autotransformations for the integrable case of the system: the former by truncating Weiss–Tabor–Carnevale expansions, and the latter by a technique of Miura transformations.

Let us consider the following system of two complex and one real nonlinear equations:

$$\begin{aligned} &\mathrm{i}a_t = a_{xx} + \alpha a^2 a^* - \mathrm{i}p \\ &p_x + \mathrm{i}\beta p + ar = 0 \\ &r_x = \tfrac{1}{2}(ap^* + a^* p) \end{aligned} \tag{1}$$

where functions a and p are complex, function r and parameters α and β are real, and * denotes the complex conjugation. This system was introduced into nonlinear optics by Doktorov and Vlasov [7], who tested its integrability by a modification [8] of the Wahlquist–Estabrook method, selected the only case $\alpha = \frac{1}{2}$ (for all β) where the prolongation structure could be closed, and constructed a Lax pair for this case (see

[9] for a detailed proof). Starting the Painlevé analysis of system (1), we must 'complexify' the system, i.e. supplement (1) by the complex conjugations of its first and second equations and consider functions $b=a^*$ and $q=p^*$ as independent of a and p respectively. Thus, we have the following system of five nonlinear equations of total order seven:

$$\begin{aligned} &a_{xx}+\alpha a^2 b-\mathrm{i}a_t-\mathrm{i}p=0\\ &b_{xx}+\alpha ab^2+\mathrm{i}b_t+\mathrm{i}q=0\\ &p_x+\mathrm{i}\beta p+ar=0\\ &q_x-\mathrm{i}\beta q+br=0\\ &r_x-\tfrac{1}{2}(aq+bp)=0. \end{aligned} \tag{2}$$

System (2) is a *normal* system written in the Kovalevskaya form, a hypersurface $\varphi(x,t)=0$ is non-characteristic for (2) if $\varphi_x\neq0$, and the general solution of (2) is determined by seven arbitrary functions given at any non-characteristic hypersurface [10]. Following Ward [11], we believe that the Painlevé property *must not* fix any structure of solutions at characteristic hypersurfaces. Analysing singularities of solutions at non-characteristic hypersurfaces $\varphi=0$, we will use the Kruskal ansatz [12] $\varphi_x=1$.

According to the Weiss–Kruskal algorithm [2] which is sensitive to algebraic and non-dominant logarithmic branch points, the leading behaviour of solutions of (2) is assumed to be algebraic: $a=a_0\varphi^{\sigma_1}+\ldots$, $b=b_0\varphi^{\sigma_2}+\ldots$, $p=p_0\varphi^{\sigma_3}+\ldots$, $q=q_0\varphi^{\sigma_4}+\ldots$, and $r=r_0\varphi^{\sigma_5}+\ldots$, where $a_0,\ldots,r_0$ are non-zero functions of t, and $\sigma_1,\ldots,\sigma_5$ are complex constants. Substituting these expressions for $a,\ldots,r$ into (2) determines the following two branches.

Branch 1. $\sigma_1=\sigma_2=-1$, $\sigma_3=\sigma_4=\sigma_5=\sigma$, $\mathrm{Re}\,\sigma>-3$, $\sigma\neq0$;
$a_0b_0=-2/\alpha$, $p_0=-a_0r_0/\sigma$, $q_0=-b_0r_0/\sigma$;
functions a_0/b_0 and r_0 are not determined, i.e. two resonances appear in zero position; the compatibility condition at these resonances is

$$\alpha=2\sigma^{-2}. \tag{3}$$

Branch 2. $\sigma_1=\tau$, $\sigma_2=-\tau-2$, $\sigma_3=\tau-2$, $\sigma_4=-\tau-4$, $\sigma_5=-3$, $\tau\neq-4,-1,2$;
$a_0b_0=(\tau-2)(\tau+4)$, $p_0=\mathrm{i}(\tau+1)(\tau+4)a_0$, $q_0=-\mathrm{i}(\tau+1)(\tau-2)b_0$, $r_0=-\mathrm{i}(\tau+1)(\tau-2)(\tau+4)$;
function a_0/b_0 is not determined, i.e. one resonance appears in zero position; the compatibility condition at this resonance is

$$\alpha=-2-20(\tau-2)^{-1}(\tau+4)^{-1}. \tag{4}$$

Since the Painlevé property bans movable branch points, constants σ and τ must be integers satisfying the condition

$$(\sigma^2+1)[(\tau+1)^2+1]=10 \tag{5}$$

which follows from (3) and (4). Solving (5) in integers, we find that there are only two values of α at which solutions of system (2) have no movable dominant algebraic branch points: (i) $\alpha=2$, $\sigma=\pm1$, $\tau=-1\pm2$, and (ii) $\alpha=\frac{1}{2}$, $\sigma=\pm2$, $\tau=-1\pm1$ (the upper or lower sign is taken for σ and τ independently).

The next step of the Weiss–Kruskal algorithm consists in finding positions of all resonances. The standard way [2, 12] brings us to the following equations determining position N of a resonance:

$$(N+1)N^2(N-3)(N-4)(N+\sigma)(N+2\sigma)=0 \tag{6}$$

for branch 1, and

$$(N+1)N(N-3)(N-6)\{N^3-7N^2+[12-10(\tau+1)^2]N+40(\tau+1)^2\}=0 \tag{7}$$

for branch 2. All roots N of (6) and (7) are integers in both cases (i) and (ii). The general solution of system (2) must contain seven arbitrary functions of t, and this is achieved in branch 1 at $\sigma=-1$ for (i) and $\sigma=-2$ for (ii) (the *generic* subbranch, where six of N are non-negative, and $N=-1$ corresponds to arbitrary φ). Second subbranch of branch 1 (positive σ) and the whole branch 2 are non-generic. So both cases $\alpha=2$ and $\alpha=\frac{1}{2}$ have passed this step of the algorithm well.

At the last step of the Weiss–Kruskal algorithm, the recursion relations are constructed, and the compatibility conditions are checked at all non-negative resonances of all branches; all the compatibility conditions must be identities. The generic subbranch ($\sigma=-1$) of the case $\alpha=2$ has resonances $N=-1, 0, 0, 1, 2, 3, 4$; however the compatibility condition $\varphi_t+2_\beta=0$ arises at resonance 1. Since the condition is not an identity, the general solution of system (2) contains a movable non-dominant logarithmic singularity when $\alpha=2$. Only the case $\alpha=\frac{1}{2}$ remains, when system (2) does possess the Painlevé property. In this case, the generic subbranch ($\sigma=-2$) has resonances $N=-1, 0, 0, 2, 3, 4, 4$, the non-generic subbranch ($\sigma=2$) of branch 1 has resonances $N=-4, -2, -1, 0, 0, 3, 4$, and branch 2 has resonances $N=-2, -1, 0, 3, 4, 5, 6$. The compatibility conditions at $N=0$, namely (3) and (4), have been satisfied. Ten more compatibility conditions at positive N turn out to be identities as well. Checking this by hand is the best way to get a feeling for how mysterious the Painlevé property is.

Thus, the singularity analysis of (2) indicates that the system has the Painlevé property only in the case $\alpha=\frac{1}{2}$ (for all β; but parameter β is unessential: if $\beta\neq 0$, then scale transformations can make $\beta=1$). This agrees with the result by Doktorov and Vlasov [7]. Hereafter we will consider only the integrable case $\alpha=\frac{1}{2}$. Since its Lax pair is already known [7], let us try to find a Bäcklund autotransformation by the method of truncating Weiss–Tabor–Carnevale expansions [13]. For this purpose, equivalent singular expansions by powers of function $\chi=(\varphi^{-1}\varphi_x-\frac{1}{2}\varphi_x^{-1}\varphi_{xx})^{-1}$, proposed by Conte [14], are very useful (here and below the Kruskal ansatz $\varphi_x=1$ is *not* used for function φ determining the singularity manifold). Truncating must be performed in the generic subbranch, because the Bäcklund autotransformation sought should be applicable to any solution. Substituting expressions $a=a_0\chi^{-1}+a_1$, $b=b_0\chi^{-1}+b_1$, $p=p_0\chi^{-2}+p_1\chi^{-1}+p_2$, $q=q_0\chi^{-2}+q_1\chi^{-1}+q_2$ and $r=r_0\chi^{-2}+r_1\chi^{-1}+r_2$ into (2) gives us a system of 20 nonlinear equations for 14 functions of x and t: $a_0, a_1, b_0, b_1, p_0, p_1, p_2, q_0, q_1, q_2, r_0, r_1, r_2$ and φ. That system turns out to be *compatible*, gives explicit expressions for coefficients $a_0, a_1, \ldots, r_1$ and r_2 in terms of two functions φ and f (f appears at resonance 0 as $a_0/b_0=\exp(2\mathrm{i}f)$) and imposes the following two nonlinear equations of third and fourth order on φ and f:

$$f_t=\tfrac{3}{2}f_x^2+\beta f_x+(\beta+\lambda)C-S+\beta\lambda+\tfrac{3}{2}\lambda^2 \tag{8}$$
$$f_{xxxx}+f_{xx}[(\beta+\lambda)(3f_x+C+\beta)+2S]+f_x[2(\beta+\lambda)C_x+S_x]+(\beta^2-\lambda^2)C_x+\lambda S_x-S_t=0$$

where λ is any complex constant (of integration), $S=\varphi_x^{-1}\varphi_{xxx}-\frac{3}{2}\varphi_x^{-2}\varphi_{xx}^2$ and $C=-\varphi_x^{-1}\varphi_t$. Explicit expressions for a and b in terms of φ and f are simple:

$$\begin{aligned} a&=(2\mathrm{i}\varphi^{-1}\varphi_x-\mathrm{i}\varphi_x^{-1}\varphi_{xx}+f_x-\lambda)\exp(\mathrm{i}f) \\ b&=(2\mathrm{i}\varphi^{-1}\varphi_x-\mathrm{i}\varphi_x^{-1}\varphi_{xx}-f_x+\lambda)\exp(-\mathrm{i}f). \end{aligned} \tag{9}$$

We omit complicated expressions for p, q and r, but they can be obtained, if necessary, by substituting (9) into the first three equations of system (3). The seventh-order system (8) is a normal system possessing the same characteristic directions and number of arbitrary functions in its general solution as does system (2) [10]. Expressions (9) together with corresponding expressions for p, q and r determine a Miura transformation of system (8) into system (2). The truncation procedure employed [13, 14] provides one more Miura transformation $(\varphi, f)\to(\tilde{a}, \tilde{b}, \tilde{p}, \tilde{q}, \tilde{r})$ between (8) and (2), where expressions for $\tilde{a}, \ldots, \tilde{r}$ are obtained from expressions for $a, \ldots, r$ by way of $\varphi\to\varphi+\gamma$ ($\gamma=$ constant) and $\gamma\to\infty$. This pair of Miura transformations generates a Bäcklund autotransformation for system (2), because one can find φ and f from *any* 'old' solution $a, \ldots, r$ and then map these φ and f into a 'new' solution $\tilde{a}, \ldots, \tilde{r}$.

Unfortunately, this Bäcklund autotransformation, valid for system (2), is useless for system (1). Indeed, conditions $b=a^*$ and $\tilde{b}=\tilde{a}^*$, imposed on (9) and corresponding expressions for $\tilde{a}$ and $\tilde{b}$, lead through (8) to

$$a=(\xi+\mathrm{i}\eta)\exp(\mathrm{i}\zeta x+\mathrm{i}\psi)$$

and

$$a=(-\xi+\mathrm{i}\eta)\exp(\mathrm{i}\zeta x+\mathrm{i}\psi)$$

where constants ξ, η and ζ and function $\psi(t)$ are real and arbitrary, $\zeta=\operatorname{Re}\lambda$. This means that most of the solutions of system (1) are transformed not into solutions of (1) but into solutions of 'complexified' system (2). The same phenomenon takes place for the nonlinear Schrödinger equation $\mathrm{i}a_t=a_{xx}+\frac{1}{2}a^2a^*$ which is a special case ($p=0$) of system (1). The 'complexified' equation, i.e. the system of

$$a_{xx}+\tfrac{1}{2}a^2b-\mathrm{i}a_t=0 \qquad \text{and} \qquad b_{xx}+\tfrac{1}{2}ab^2+\mathrm{i}b_t=0$$

has the Painlevé property, and the truncating procedure is compatible for it [15]. The truncated expansions have form (9) (where φ and f satisfy equations $f_x+C+\lambda=0$ and $f_t=\frac{3}{2}f_x^2-\lambda f_x-S+\frac{1}{2}\lambda^2$, notations are the same as in (8)) and represent a pair of Miura transformations which generate a Bäcklund autotransformation for the 'complexified' equation. It is surprising that this autotransformation is inapplicable to the nonlinear Schrödinger equation itself, because conditions $b=a^*$ and $\tilde{b}=\tilde{a}^*$ restrict a and $\tilde{a}$ very considerably (admissible a and $\tilde{a}$ have the same form as in the case of system (1) above, but $\psi(t)=[\zeta^2-\frac{1}{2}(\xi^2+\eta^2)]t+\text{constant}$). However, the nonlinear Schrödinger equation is known to possess another Bäcklund autotransformation, compatible with the complex conjugateness of a and a^*, which was derived from the Lax pair by Chen [16]. Therefore we may expect that Doktorov–Vlasov equations (1) possess a 'good' Bäcklund autotransformation too.

The two seemingly different approaches to constructing Bäcklund autotransformations, the methods by Chen [16] and by Weiss [13], are actually based on one idea: two *different* Miura transformations, which both map an auxiliary system into the investigated system, generate a Bäcklund autotransformation for the investigated system. The Weiss approach determines such a pair of Miura transformations by truncating singular expansions, whereas the Chen approach derives it from a Lax pair (see also

[17]). The two ways may lead to different results, as it happens for the nonlinear Schrödinger equation. However, Miura transformations can be worked with irrespectively of the Painlevé property and Lax pairs (see e.g. [18] and references therein). A procedure of finding admissible Miura transformations for nonlinear systems in the Kovalevskaya form is analogous with the procedure for scalar evolution equations, and we will describe it elsewhere. The result, obtained by very hard calculations, is as follows. The auxiliary system

$$
\begin{aligned}
&\mathrm{i}u_t = u_{xx} + (u|u_x|^2 + \tfrac{1}{2}u^* u_x^2)(\lambda - |u|^2)^{-1} + \tfrac{1}{8}u|u|^2 - \mathrm{i}v(\lambda - |u|^2)^{1/2} \\
&v_x + \tfrac{1}{2}u(u_x v^* - u_x^* v)(\lambda - |u|^2)^{-1} + \mathrm{i}\beta v + uw = 0 \\
&w_x = \tfrac{1}{2}\mathrm{i}\beta(u_x v^* - u_x^* v)(\lambda - |u|^2)^{-1} + \tfrac{1}{8}(uv^* + u^* v)
\end{aligned}
\tag{10}
$$

is mapped into system (1) by the following pair of Miura transformations (the upper or lower sign is taken for all expressions simultaneously):

$$
\begin{aligned}
&a = u_x(\lambda - |u|^2)^{-1} \pm \tfrac{1}{2}u \\
&p = -uw - \mathrm{i}\beta v \pm \tfrac{1}{2}v(\lambda - |u|^2)^{1/2} \\
&r = w(\lambda - |u|^2)^{1/2} \pm \tfrac{1}{4}(uv^* + u^* v)
\end{aligned}
\tag{11}
$$

where functions u and v are complex, function w is real, and λ is any real constant. The pair (11) and system (10) generate a Bäcklund autotransformation for system (1) (and for 'complexified' system (2) as well, if one supplements (10) and (11) by complex conjugations and considers u^* and v^* as independent of u and v). This Bäcklund autotransformation seems to be derivable from the Lax pair [7, 9] of Doktorov–Vlasov equations (1) by the Chen method [16]. Indeed, taking $a = u_x(\lambda - |u|^2)^{-1} + \frac{1}{2}u$ and $\tilde{a} = u_x(\lambda - |u|^2)^{-1} - \frac{1}{2}u$ from (11) and removing u from these expressions, we get one simple relation between the a components of 'old' and 'new' solutions of (1): $a_x - \tilde{a}_x = \frac{1}{2}(a + \tilde{a})(\lambda - |a - a|^2)$ (however, relations between other components are complicated and hardly useful). The Chen method [16] gives the same relation for the nonlinear Schrödinger equation $\mathrm{i}a_t = a_{xx} + \frac{1}{2}a^2 a^*$ which has the same L-operator as the Doktorov–Vlasov equations have.

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